

ERRATA: SCATTERING THRESHOLD FOR THE FOCUSING NONLINEAR KLEIN-GORDON EQUATION

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ABSTRACT. This article resolves some errors in the paper “Scattering threshold for the focusing nonlinear Klein-Gordon equation”, Analysis & PDE **4** (2011) no. 3, 405–460. The errors are in the energy-critical cases in two and higher dimensions.

1. THE ERRORS AND THE MISSING INGREDIENT

This article resolves some errors in [1]. One correction affects also [2, 3]. The section and equation numbers etc. in [1] will be underlined for distinction. The major errors are the following three: one in Section 2 for the existence of mass-shifted ground state in the two dimensional energy-critical case, and two in Section 5 for the nonlinear profile decomposition in the higher dimensional energy-critical case.

- (1) In the proof of Lemma 2.6, it is not precluded that the weak limit Q in (2-67) is zero. Hence the existence of Q in the case $c \leq 1$ is not proved.
- (2) In (5-56), we do not have $\|\vec{V}_n(\tau_n) - \vec{V}_\infty(\tau_n)\|_{L_x^2} \rightarrow 0$ when $h_\infty = 0$, $\tau_\infty = \pm\infty$ and $\liminf_{n \rightarrow \infty} |\tau_n h_n^2| > 0$. Indeed, assuming that $\tau_n h_n^2 \rightarrow m \in [-\infty, \infty]$ after extraction of a subsequence, we have

$$\|\vec{V}_n(\tau_n) - \vec{V}_\infty(\tau_n)\|_{L_x^2} \rightarrow \begin{cases} \|(e^{im/(2|\nabla|)} - 1)\psi\|_{L_x^2} & (|m| < \infty), \\ \sqrt{2}\|\psi\|_{L_x^2} & (m = \pm\infty). \end{cases} \quad (1.1)$$

- (3) In the proof of Lemma 5.6, the global bound (5-96) does not follow from the uniform bound on finite time intervals, since the required largeness of n depends on the size of the interval I .

(1) is concerned only with a very critical case of exponential nonlinearity in two dimensions $d = 2$. More precisely, it is problematic only if

$$0 < \limsup_{|u| \rightarrow \infty} e^{-\kappa_0|u|^2} |u|^2 f(u) < \infty, \quad (1.2)$$

where κ_0 is the exponent in (1-29). (2)–(3) are crucial only in the H^1 critical case of higher dimensions $d \geq 3$, with $h_\infty = 0$: the concentration by scaling in the nonlinear profile, where we need to modify the definition of the nonlinear concentrating waves, and then solve the massless limit problem for NLKG (see Theorem 3.1 below). In the other case, i.e. with the subcritical or exponential nonlinearity or with $h_\infty = 1$, we still need to take care of (3), but it is rather superficial change.

2. CORRECTION FOR (1)

We do not know if [Lemma 2.6](#) holds true in the very critical case (1.2). So we add the following assumption

$$\limsup_{|u| \rightarrow \infty} e^{-\kappa_0|u|^2} |u|^2 f(u) \in \{0, \infty\} \quad (2.1)$$

in [Proposition 1.2\(3\)](#) and in [Lemma 2.6](#). The existence of Q was used in [1] only to characterize the threshold energy m , so the rest of the paper is not affected by it.

In [2, (1.24)], the existence of Q is mentioned to characterize the threshold $m^{(c)}$. It should be also restricted by (2.1), but the rest of the paper [2] does not really need Q . Removing Q , [2, (2.3)] should be replaced with

$$m \leq H_p^{(c)}(\varphi), \quad (2.2)$$

[2, (2.6)] should be replaced with

$$m \leq J^{(c)}(\lambda\varphi) = H_p^{(c)}(\lambda\varphi) \leq H_p^{(c)}(\varphi), \quad (2.3)$$

and [2, (2.7)] with

$$\begin{aligned} \ddot{y} &= (2+p)\|\dot{u}\|_{L^2}^2 + 2p(H_p^{(1)}(u) - m) \\ &= (4+\varepsilon)\|\dot{u}\|_{L^2}^2 + (1-c)\varepsilon\|u\|_{L^2}^2 + 2p(H_p^{(c)}(u) - m) \\ &\geq (1+\varepsilon/4)\dot{y}^2/y + (1-c)\varepsilon y. \end{aligned} \quad (2.4)$$

The existence of Q is also mentioned in [3, Theorem 5.1]. It should be also restricted by (2.1). The rest of the paper [3] remains unaffected.

We still need to prove [Lemma 2.6](#) under the new restriction (2.1). If the limit (2.1) is infinite, then [3, Theorem 1.5(B)] implies $C_{\text{TM}}^*(F) = \infty > 1$. In this case, the proof of [Lemma 2.6](#) remains valid. If the limit (2.1) is zero, then [3, Theorem 1.5(B)] implies $C_{\text{TM}}^*(F) < \infty$. In this case, we do not argue as in [1], but rely on the compactness [3, Theorem 1.5(C)]. Let $\varphi_n \in H^1(\mathbb{R}^2)$ be a normalized maximizing sequence for $C_{\text{TM}}^*(F)$, i.e.

$$\|\varphi_n\|_{L^2} = 1, \quad \kappa_0 \|\nabla \varphi_n\|_{L^2}^2 \leq 4\pi, \quad 2F(\varphi_n) \rightarrow C := C_{\text{TM}}^*(F) \in (0, \infty). \quad (2.5)$$

By the standard rearrangement, and the H^1 boundedness, we may assume that φ_n are radially decreasing and $\varphi_n \rightarrow \exists \varphi$ weakly in $H^1(\mathbb{R}^2)$. By [3, Theorem 1.5(C)], we have $2F(\varphi_n) \rightarrow 2F(\varphi) = C > 0$. In particular, $\varphi \neq 0$. Since $\kappa_0 \|\nabla \varphi\|_{L^2}^2 \leq 4\pi$ and $\|\varphi\|_{L^2} \leq 1$ by the weak convergence, we deduce from the definition of $C_{\text{TM}}^*(F)$ that $\|\varphi\|_{L^2} = 1$ and φ is a maximizer. Hence for a Lagrange multiplier $\mu \geq 0$,

$$f'(\varphi) - C\varphi = -\mu\Delta\varphi. \quad (2.6)$$

$\mu \neq 0$ is obvious by the decay order of f' as $\varphi \rightarrow 0$. Hence $\mu > 0$ and so $\kappa_0 \|\nabla \varphi\|_{L^2}^2 = 4\pi$, since otherwise we could increase both $F(\varphi)$ and $\|\nabla \varphi\|_{L^2}^2$ by the L^2 scaling $\varphi_{1,-1}^\lambda$ with $\lambda > 0$, using the L^2 super-critical condition [\(1-21\)](#). Then $Q(x) := \varphi(\mu^{-1/2}x) \in H^2(\mathbb{R}^2)$ satisfies

$$-\Delta Q + CQ = f'(Q), \quad \kappa_0 \|\nabla Q\|_{L^2}^2 = 4\pi, \quad 2F(Q) = C\|Q\|_{L^2}^2, \quad (2.7)$$

Hence $J^{(C)}(Q) = \frac{1}{2}\|\nabla Q\|_{L^2}^2 = 2\pi/\kappa_0$. The rest of the proof of [Lemma 2.6](#), namely the proof of $m_{\alpha,\beta} = m_{0,1} = 2\pi/\kappa_0$ remains valid.

3. CORRECTION FOR (2)-(3)

For (2)-(3), we do not have to modify the main results, but need to correct the proof, including the definition of the nonlinear profile decomposition. Henceforth, we always assume that $0 < h_n \rightarrow h_\infty$, $(t_n, x_n) \in \mathbb{R}^{1+d}$, and $\tau_n = -t_n/h_n \rightarrow \tau_\infty \in [-\infty, \infty]$ are sequences. The main problematic case is when the energy concentrates, namely $h_\infty = 0$, which can happen only in the energy critical case [\(1-28\)](#):

$$d \geq 3, \quad f(u) = |u|^{2^*}/2^*, \quad 2^* = 2d/(d-2). \quad (3.1)$$

First we modify the vector notation in [\(4-1\)](#). For any real-valued function $a(t, x)$, the complex-valued functions $\vec{a}, \bar{a}, \bar{\bar{a}}$ are defined by

$$\vec{a} := (\langle \nabla \rangle - i\partial_t)a, \quad \bar{a} := (\langle \nabla \rangle_n - i\partial_t)a, \quad \bar{\bar{a}} := (\langle \nabla \rangle_\infty - i\partial_t)a, \quad (3.2)$$

where $\langle \nabla \rangle_* = \sqrt{h_*^2 - \Delta}$ as in [\(5-1\)](#). Hence a is recovered from either of them by

$$a = \operatorname{Re} \langle \nabla \rangle^{-1} \vec{a} = \operatorname{Re} \langle \nabla \rangle_n^{-1} \bar{a} = \operatorname{Re} \langle \nabla \rangle_\infty^{-1} \bar{\bar{a}}. \quad (3.3)$$

Note that (\bar{a}, a) was denoted by (\vec{a}, \hat{a}) in [1], but it was confusing. Indeed, $u_{(n)}$ in [\(5-55\)](#) did not make sense if $h_\infty = 0$, since $\vec{u}_{(n)}$ in [\(5-54\)](#) was not in the form [\(4-1\)](#). So we replace [\(5-54\)](#) with

$$\vec{u}_{(n)} = T_n \vec{U}_{(n)}((t - t_n)/h_n), \quad (3.4)$$

where $\vec{U}_{(n)}$ is defined by

$$\vec{V}_n := e^{it\langle \nabla \rangle_n} \psi, \quad \vec{U}_{(n)} = \vec{V}_n - i \int_{\tau_\infty}^t e^{i(t-s)\langle \nabla \rangle_n} f'(U_{(n)}) ds. \quad (3.5)$$

Then $u_{(n)} = h_n T_n U_{(n)}((t - t_n)/h_n)$ is a solution of NLKG satisfying

$$\lim_{t \rightarrow \tau_\infty} \|(\vec{u}_{(n)} - \vec{v}_n)(th_n + t_n)\|_{L_x^2} = 0. \quad (3.6)$$

In other words, we keep NLKG in defining the profiles, even if $h_\infty = 0$. Note that if $h_\infty = 1$ then $\vec{U}_{(n)} = \vec{U}_\infty$ and so $u_{(n)}$ is unchanged.

By the change of [\(5-54\)](#) to (3.4), the problematic [\(5-56\)](#) is replaced with

$$\|\vec{u}_n(0) - \vec{u}_{(n)}(0)\|_{L_x^2} = \left\| \int_{\tau_\infty h_n + t_n}^0 (= \tau_n h_n + t_n) e^{-is\langle \nabla \rangle} f'(u_{(n)}) ds \right\|_{L_x^2} \rightarrow 0. \quad (3.7)$$

In order to prove the last limit, as well as the global Strichartz approximation for (3), we need the convergence in the massless limit of the H^1 critical NLKG:

Theorem 3.1. *Assume [\(1-28\)](#) and $h_\infty = 0$. Let \vec{U}_∞ be the solution of*

$$\vec{V}_\infty := e^{it|\nabla|} \psi, \quad \vec{U}_\infty = \vec{V}_\infty - i \int_{\tau_\infty}^t e^{i(t-s)|\nabla|} f'(U_\infty) ds. \quad (3.8)$$

Let $\vec{U}_{(n)}$ be the solution of (3.5) and $\vec{u}_{(n)}(t) := T_n \vec{U}_{(n)}((t - t_n)/h_n)$. Suppose that $U_\infty \in [W]_2^\bullet(J)$ for some interval J whose closure in $[-\infty, \infty]$ contains τ_∞ . Then for any bounded subinterval $I \subset J$, we have, as $n \rightarrow \infty$,

$$\begin{aligned} \|\vec{U}_{(n)} - \vec{U}_\infty\|_{L_{t \in I}^\infty L_x^2} + \|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + \|u_{(n)}\|_{[W]_0(J)} &\rightarrow 0, \\ \|u_{(n)}\|_{([W]_2 \cap [M]_0)(h_n J + t_n)} &\sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + o(1). \end{aligned} \quad (3.9)$$

Postponing the proof of the above theorem to the next section, we continue to correct Section 5. (3.7) in the case of $h_\infty = 0$ follows from the above estimate and $\tau_n \rightarrow \tau_\infty$ via Strichartz:

$$\begin{aligned} \left\| \int_{\tau_\infty h_n + t_n}^0 e^{-is\langle \nabla \rangle} f'(u_{(n)}) ds \right\|_{L_x^2} &\lesssim \|f'(u_{(n)})\|_{[W^{*(1)}]_2(I_n)} \\ &\lesssim \|u_{(n)}\|_{([W]_2 \cap [M]_0)(I_n)}^{2^*-1} \lesssim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J_n)}^{2^*-1} + o(1) = o(1), \end{aligned} \quad (3.10)$$

where $I_n := (0, \tau_\infty h_n + t_n) \cup (\tau_\infty h_n + t_n, 0)$ and $J_n := (\tau_n, \tau_\infty) \cup (\tau_\infty, \tau_n)$.

We modify the definition of ST in (5-59)–(5-60) in the \dot{H}^1 critical case (1-28) to

$$ST = [W]_2, \quad ST^* = [W^{*(1)}]_2 + L_t^1 L_x^2, \quad ST_\infty^\diamond := \begin{cases} [W]_2 & (h_\infty^\diamond = 1), \\ [W]_2^\bullet & (h_\infty^\diamond = 0). \end{cases} \quad (3.11)$$

Indeed, $[K]_2$ and $[K^{*(1)}]_2$ norms are not needed in the \dot{H}^1 critical case. Then we simply discard the estimates (5-61)–(5-62).

Next we reprove Lemma 5.5, extending it to unbounded intervals I . The above theorem implies that we can replace (5-64) with the stronger¹

$$\limsup_{n \rightarrow \infty} \|u_{(n)}^j\|_{ST(\mathbb{R})} \lesssim \|U_\infty^j\|_{ST_\infty^j(\mathbb{R})}, \quad (3.12)$$

if $h_\infty^j = 0$, while it is trivial if $h_\infty^j = 1$. The proof of (5-65) for $h_\infty^j = 1$ did not use the boundedness of I , so we may assume that all h_∞^j are 0. Then the above theorem implies that $\|u_{(n)}^{<k}\|_{[W]_0(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$, so it suffices to estimate the homogeneous norm $[W]_2^\bullet(\mathbb{R})$. We have

$$\|u_{(n)}^{<k}\|_{[W]_2^\bullet(\mathbb{R})} \sim \sum_{l=1}^d \left\| \sum_{j < k} \check{u}_{n,m}^{j,l} \right\|_{L_t^p \ell_m^2 \mathbb{Z} L_x^q} \quad (3.13)$$

with $(1/p, 1/q, s) = W$ and

$$\check{u}_{n,m}^{j,l} := 2^{sm} \delta_m^l h_n^j T_n^j U_{(n)}^j((t - t_n^j)/h_n^j). \quad (3.14)$$

Defining $\check{u}_{n,m,R}^{j,l}$ by (5-77), we have

$$\|\check{u}_{n,m}^{j,l} - \check{u}_{n,m,R}^{j,l}\|_{L_t^p \ell_m^2 L_x^q} \lesssim \|2^{sm} \delta_m^l U_{(n)}^j\|_{L_t^p \ell_m^2 L_x^q(|t|+|m|+|x|>R)} \rightarrow 0, \quad (R \rightarrow \infty) \quad (3.15)$$

which is still uniform in n , since by the above theorem $U_{(n)}^j$ is approximated by U_∞^j in $[W]_2^\bullet(\mathbb{R})$, which is equivalent to the last norm without the restriction by R . Thus we obtain (5-65) by the disjoint support property for large n .

¹Recall that \widehat{U}_∞^j in [1] is denoted by U_∞^j in this errata according to (3.2).

According to the change of $u_{(n)}^j$, we replace the nonlinear decomposition (5-66) with a simpler form:

$$\lim_{n \rightarrow \infty} \|f'(u_{(n)}^{<k}) - \sum_{j < k} f'(u_{(n)}^j)\|_{ST^*(I)} = 0, \quad (3.16)$$

which is the same as (5-66) if $h_\infty^j = 1$. In that case, however, we used that I was bounded in (5-82). We replace it with an interpolation between (4-84) and

$$\|f'_S(u)\|_{[(1-\theta_0)K + \theta_0 W]^{*(1)}_2(I)} \lesssim \|u\|_{[K]_2(I)} \|u\|_{[K]_0(I)}^{p_1} \lesssim \|u\|_{[K]_2(I)}^{p_1+1}, \quad (3.17)$$

where we can choose some $\theta_0 \in (0, 1)$ since $p_1 > 4/d$ (and choosing p_1 close enough to $4/d$ if necessary). Since $Z := ((1 - \theta_0)K + \theta_0 W)^{*(1)}$ is an interior dual-admissible exponent, we can find some $\theta_1 \in (0, 1)$ such that $\theta_1 Y + (1 - \theta_1)Z$ is also a dual-admissible exponent. Interpolating (3.17) with (4-84), we have

$$\|f'_S(u) - f'_S(v)\|_{[\theta_1 Y + (1-\theta_1)Z]_2(I)} \lesssim \|(u, v)\|_{[K]_2(I) \cap [Q]_{2p_1}(I)}^{p_1+1-\theta_1} \|u - v\|_{[P]_2(I)}^{\theta_1}. \quad (3.18)$$

Thus we obtain (5-66) on any subset I in the subcritical/exponential cases. In the \dot{H}^1 critical case (1-28), we discard $u_{(n)}^j$ in (5-85) and prove (3.16) directly, putting

$$U_{n,R}^j(t, x) := \chi_R(t, x) U_{(n)}^j(t, x) \times \prod \{(1 - \chi_{h_n^{j,l} R})(t - t_n^{j,l}, x - x_n^{j,l}) \mid 1 \leq l < k, h_n^l R < h_n^j\}. \quad (3.19)$$

It is still uniformly bounded in $([H]_2^\bullet \cap [W]_2^\bullet)(\mathbb{R})$, and $U_{n,R}^j - \chi_R U_{(n)}^j \rightarrow 0$ in $[M]_0(\mathbb{R})$ as $n \rightarrow \infty$, thanks to the above theorem, as well as in $[L]_0$, and also $\chi_R U_{(n)}^j \rightarrow U_{(n)}^j$ as $R \rightarrow \infty$. Hence we may replace $u_{(n)}^j$ in (3.16) by $u_{(n),R}^j := h_n^j T_n^j U_{n,R}^j((t - t_n^j)/h_n^j)$, using (4-62) for $d \leq 5$, and a similar interpolation argument as above for $d \geq 6$, see (4.17)–(4.20) below. Then we obtain (3.16) by the disjoint support property, in the same way as (5-94).

With the above corrections, now we reprove Lemma 5.6. First, (5-100) holds for any subset $I \subset \mathbb{R}$, by the above improvement of Lemma 5.5. Now, thanks to the change of $u_{(n)}^j$, (5-101) is simplified to

$$eq(u_{(n)}^{<k}) = f'(u_{(n)}^{<k}) - \sum_{j < k} f'(u_{(n)}^j), \quad (3.20)$$

which is vanishing by (3.16). Hence we obtain (5-103). We also obtain (5-104) on \mathbb{R} by the same nonlinear estimates as we used above. Then applying Lemma 4.5 on \mathbb{R} , we obtain the desired Lemma 5.6.

Section 6 is almost unchanged, except for the obvious modification in (6-6) due to the change of $u_{(n)}$, namely

$$\vec{u}_{(n)}^j = T_n^j \vec{U}_{(n)}^j((t - t_n^j)/h_n^j), \quad (3.21)$$

and the notational change in (6-7)–(6-9) from $(\vec{U}_\infty^0, \widehat{U}_\infty^0)$ to $(\vec{U}_\infty^0, U_\infty^0)$ due to (3.2). Since the case $h_\infty = 0$ is eliminated in the proof of Lemma 6.1, the errors (2)–(3) do not affect the rest of the paper.

4. MASSLESS LIMIT OF SCATTERING FOR THE CRITICAL NLKG

It remains to prove Theorem 3.1. Throughout this section, we assume (1-28). The main idea is to decompose the time interval into a bounded subinterval and neighborhoods of $\pm\infty$. On the bounded part, we have strong convergence in the massless limit. In the neighborhoods of $t = \pm\infty$, we do not have strong convergence, but the Strichartz norms are uniformly controlled via the asymptotic free profiles.

The first ingredient concerns the uniform Strichartz bound for free waves.

Lemma 4.1. *Let $\vec{v}_n = e^{it\langle\nabla\rangle}T_n\psi$, $h_\infty = 0$, $\vec{V}_\infty = e^{it|\nabla|}\psi$, and let $Z \in [0, 1/2] \times [0, 1/2] \times [0, 1)$ satisfy $\text{reg}^0(Z) = 1$ and $\text{str}^0(Z) \leq 0$, namely a wave-admissible Strichartz exponent except for the energy norm. Then we have*

$$\limsup_{n \rightarrow \infty} \|v_n\|_{[Z]_2(0,\infty)} \lesssim \|V_\infty\|_{[Z]_2^\bullet(0,\infty)}, \quad \lim_{n \rightarrow \infty} \|P_{<1}v_n\|_{[Z]_2(0,\infty)} = 0, \quad (4.1)$$

where $P_{<a}$ denotes the smooth cut-off for the Fourier region $|\xi| < 2a$ defined by $P_{<a}\varphi = a^d \Lambda_0(ax) * \varphi$, with $\Lambda_0 \in \mathcal{S}(\mathbb{R}^d)$ in the proof of [Lemma 5.1](#). If $Z_3 = 0$, then we have also $\|v_n\|_{[Z]_0(0,\infty)} \rightarrow \|V_\infty\|_{[Z]_0(0,\infty)}$.

Proof. Let $\vec{v}_n(t) = T_n \vec{V}_n(t/h_n)$. The Strichartz estimate for the Klein-Gordon and the wave equations

$$\|v_n\|_{[Z]_2(0,\infty)} \lesssim \|T_n\psi\|_{L^2} = \|\psi\|_{L^2}, \quad \|V_\infty\|_{[Z]_2^\bullet(0,\infty)} \lesssim \|\psi\|_{L^2} \quad (4.2)$$

implies that it suffices to consider ψ in a dense subset of $L^2(\mathbb{R}^d)$. Hence we may assume that $\mathcal{F}\psi$ is C^∞ with a compact supp $\mathcal{F}\psi \not\equiv 0$. Since $0 < \langle\xi\rangle_n - \langle\xi\rangle_\infty \leq h_n^2/|\xi|$,

$$|(e^{it\langle\xi\rangle_n} \langle\xi\rangle_n^{-1} - e^{it|\xi|} |\xi|^{-1})| \lesssim |t| h_n^2 |\xi|^{-2} + h_n^2 |\xi|^{-3}, \quad (4.3)$$

and so, under the above assumption on ψ , for any $s \in \mathbb{R}$, and any sequence $S_n > 0$,

$$\|V_n - V_\infty\|_{L^\infty(0,S_n;H^s)} \leq \langle S_n \rangle h_n^2 C(s, \psi). \quad (4.4)$$

Hence by Sobolev in x and Hölder in t ,

$$\|V_n - V_\infty\|_{([Z]_2^\bullet \cap [Z]_0)(0,S_n)} \leq \langle S_n \rangle^{1+Z_1} h_n^2 C(s, \psi). \quad (4.5)$$

We deduce that if $S_n \rightarrow \infty$ and $S_n^{1+Z_1} h_n^2 \rightarrow 0$, then using the (approximate) scale invariance of $[Z]_2^\bullet$,

$$\begin{aligned} \|v_n\|_{[Z]_2(0,h_n S_n)} &\sim \|v_n\|_{[Z]_2^\bullet(0,h_n S_n)} + \|P_{<1}v_n\|_{[Z]_0(0,h_n S_n)}, \\ \|v_n\|_{[Z]_2^\bullet(0,h_n S_n)} &\sim \|V_n\|_{[Z]_2^\bullet(0,S_n)} \rightarrow \|V_\infty\|_{[Z]_2^\bullet(0,\infty)}, \\ \|P_{<1}v_n\|_{[Z]_0(0,h_n S_n)} &\sim \|h_n^{Z_3} P_{<h_n} V_n\|_{[Z]_0(0,S_n)} \rightarrow 0, \end{aligned} \quad (4.6)$$

and similarly if $Z_3 = 0$, $\|v_n\|_{[Z]_0(0,h_n S_n)} = \|V_n\|_{[Z]_0(0,S_n)} \rightarrow \|V_\infty\|_{[Z]_0(0,\infty)}$.

Next, the dispersive decay of wave-type for the Klein-Gordon equation

$$\|e^{it\langle\nabla\rangle}\varphi\|_{B_{q,2}^0} \lesssim |t|^{-(d-1)\alpha} \|\varphi\|_{B_{q',2}^s} \quad \alpha := \frac{1}{2} - \frac{1}{q} \in [0, 1/2], \quad s := (d+1)\alpha, \quad (4.7)$$

together with the embedding $L^{q'} \subset B_{q',2}^0$ implies that

$$\begin{aligned} \|v_n(t)\|_{B_{q,2}^\sigma} &\lesssim |t|^{-(d-1)\alpha} \|\langle\nabla\rangle^{\sigma+s-1} T_n\psi\|_{L^{q'}} \\ &= |t|^{-(d-1)\alpha} h_n^{1-\alpha-\sigma} \|\langle\nabla\rangle_n^{\sigma+s-1} \psi\|_{L^{q'}}, \end{aligned} \quad (4.8)$$

and so, putting $\alpha = 1/2 - Z_2$,

$$\begin{aligned} \|v_n\|_{[Z]_2(h_n S_n, \infty)} &\leq C(\psi) h_n^{1-\alpha-Z_3} \|t^{-(d-1)\alpha}\|_{L_t^{1/Z_1}(h_n S_n, \infty)} \\ &\sim C(\psi) h_n^{1-\alpha-Z_3} (h_n S_n)^{Z_1-(d-1)\alpha} = C(\psi) S_n^{\alpha-1+Z_3} \rightarrow 0 \end{aligned} \quad (4.9)$$

where we used that $\text{reg}^0(Z) = Z_3 - Z_1 + d\alpha = 1$ in the last identity, and

$$\alpha - 1 + Z_3 = \text{reg}^0(Z) + \text{str}^0(Z) - 1 - Z_1 < 0 \quad (4.10)$$

in taking the limit. Note that the above exponent is zero at the energy space $Z = (0, 1/2, 1)$, which is excluded by the assumption. The estimate in $[Z]_0(h_n S_n, \infty)$ for $Z_3 = 0$ is done in the same way. Combining them with the above estimates on $(0, h_n S_n)$ leads to the conclusion via the density argument. \square

The second ingredient is convergence or propagation of small disturbance on finite intervals, which is uniformly controlled by the Strichartz norm of U_∞ .

Lemma 4.2. *For any $0 < M, \varepsilon < \infty$, there exists $\delta = \delta(\varepsilon, M) \in (0, 1)$ with the following property. Let $h_\infty = 0$ and let U_∞ be a solution of NLW on some interval J satisfying $\|U_\infty\|_{([H]_2^\bullet \cap [W]_2^\bullet)(J)} \leq M$. Then for any bounded subinterval $I \subset J$ with $0 \in I$ and any $\varphi_n \in L^2(\mathbb{R}^d)$ with $\|\varphi_n\|_{L^2} < \delta$, the unique solution U_n of*

$$(\partial_t^2 - \Delta + h_n^2)U_n = f'(U_n), \quad \vec{U}_n(0) = \vec{U}_\infty(0) + \varphi_n \quad (4.11)$$

exists on I for large n , satisfying

$$\|\vec{U}_n - \vec{U}_\infty\|_{L_t^\infty L_x^2(I)} + \|U_n - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(I)} < \varepsilon, \quad (4.12)$$

and $\|h_n T_n U_n((t - t_n)/h_n)\|_{[W]_0(h_n I + t_n)} \lesssim \delta$ for large n .

Proof. We give the detail only in the harder case $d \geq 6$, where we need the exotic Strichartz norms. Let $\gamma_n := U_n - U_\infty$ and $\vec{\gamma}_n := \vec{U}_n - \vec{U}_\infty$, then

$$(\partial_t^2 - \Delta)\gamma_n = f'(U_\infty + \gamma_n) - f'(U_\infty) - h_n^2 U_n. \quad (4.13)$$

Remark however that $\vec{\gamma}_n$ is not written only by γ_n . It suffices to prove the following

Claim. There exist constants $\theta \in (0, 1)$ and $C > 1$ such that if

$$\|U_\infty\|_{([W]_2^\bullet \cap [\widetilde{M}]_{2p}^\bullet)(0, S)} \leq \eta, \quad \|\vec{\gamma}_n(0)\|_{L^2} \ll 1 \quad (4.14)$$

for some $0 < S < \infty$ and $0 < \eta \ll 1$, where $p = 2^* - 2 = 4/(d - 2)$, then

$$\|\vec{\gamma}_n\|_{L_t^\infty(0, S; L_x^2)} + \|\gamma_n\|_{[W]_2^\bullet(0, S)} \leq C[\|\vec{\gamma}_n(0)\|_{L^2} + \|\vec{\gamma}_n(0)\|_{L^2}^\theta \eta^{(p+1)(1-\theta)}]. \quad (4.15)$$

Proof of the claim. The exotic Strichartz estimate for the wave equation yields on the time interval $(0, S)$

$$\|\gamma_n\|_{[\widetilde{N}]_2^\bullet} \lesssim \|\vec{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[Y]_2} + \|h_n^2 U_n\|_{L_t^1 L_x^2}, \quad (4.16)$$

while the nonlinear estimate in the Besov space yields

$$\begin{aligned} &\|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[Y]_2} \\ &\lesssim \|(U_\infty, \gamma_n)\|_{[M]_0}^p \|\gamma_n\|_{[\widetilde{N}]_2^\bullet} + \|(U_\infty, \gamma_n)\|_{[\widetilde{M}]_{2p}^\bullet}^p \|\gamma_n\|_{[N]_0}, \end{aligned} \quad (4.17)$$

and we have $\|\vec{\gamma}_n(0)\|_{L^2} \lesssim \|\vec{\gamma}_n(0)\|_{L^2} + o(1)$. The $L_t^1 L_x^2$ norm is estimated by

$$\|h_n^2 U_n\|_{L_t^1 L_x^2} \leq \|h_n \vec{U}_n\|_{L_t^1 L_x^2} \leq h_n S \|\vec{\gamma}_n + \vec{U}_\infty\|_{L_t^\infty L_x^2}. \quad (4.18)$$

Define $\underline{W}, O \in [0, 1/2]^3$ by

$$\begin{aligned}\underline{W} &:= W - \frac{1}{2}(0, 1/d, 1) = \left(\frac{d-1}{2(d+1)}, \frac{d^2-2d-1}{2d(d+1)}, 0\right), \\ O &:= W + p\underline{W} = \left(\frac{(d+2)(d-1)}{2(d+1)(d-2)}, \frac{d^3+d^2-6d-4}{2(d-2)d(d+1)}, 1/2\right).\end{aligned}\tag{4.19}$$

Then O is an interior dual exponent of the standard Strichartz, and so, there is small $\theta \in (0, 1)$ such that $\theta Y + (1-\theta)O$ is also a dual exponent. Hence the standard Strichartz yields for any wave-admissible exponent Z ,

$$\begin{aligned}\|\gamma_n\|_{[Z]_2^\bullet} + \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2} \\ \lesssim \|\tilde{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[\theta Y + (1-\theta)O]_2^\bullet} + \|h_n^2 U_n\|_{L_t^1 L_x^2},\end{aligned}\tag{4.20}$$

where the nonlinear part is already estimated in $[Y]_2^\bullet$, while

$$\|f'(U_\infty + \gamma_n)\|_{[O]_2^\bullet} + \|f'(U_\infty)\|_{[O]_2^\bullet} \lesssim \eta^{p+1} + \|\gamma_n\|_{[W]_2^\bullet}^{p+1}.\tag{4.21}$$

Hence we have

$$\begin{aligned}\|\gamma_n\|_{[\tilde{N}]_2^\bullet} &\lesssim \|\tilde{\gamma}_n(0)\|_{L^2} + A + B, \\ \|\gamma_n\|_{[W]_2^\bullet \cap [\tilde{M}]_{2p}^\bullet} + \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2} &\lesssim \|\tilde{\gamma}_n(0)\|_{L^2} + A^\theta (\eta + \|\gamma_n\|_{[W]_2^\bullet})^{(1-\theta)(p+1)} + B, \\ A &\lesssim (\eta + \|\gamma_n\|_{[\tilde{M}]_{2p}^\bullet})^p \|\gamma_n\|_{[\tilde{N}]_2^\bullet}, \quad B \lesssim S h_n \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2} + o(1).\end{aligned}\tag{4.22}$$

Assuming that $\|\gamma_n\|_{[\tilde{M}]_{2p}^\bullet} \ll 1$ and that $\|\tilde{\gamma}_n\|_{L_t^\infty L_x^2}$ is bounded in n , we deduce from the above estimates that

$$\begin{aligned}A &\ll \|\gamma_n\|_{[\tilde{N}]_2^\bullet} \lesssim \|\tilde{\gamma}_n(0)\|_{L^2} + o(1), \quad B = o(1), \\ \|\gamma_n\|_{[W]_2^\bullet \cap [\tilde{M}]_{2p}^\bullet} + \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2} &\lesssim \|\tilde{\gamma}_n(0)\|_{L^2} + \|\tilde{\gamma}_n(0)\|_{L^2}^\theta \eta^{(1-\theta)(p+1)} + o(1).\end{aligned}\tag{4.23}$$

It remains to prove the uniform bound on $\|\tilde{\gamma}_n\|_{L_t^\infty L_x^2}$. Let V_∞, V_n, v_n be the free solutions defined by

$$\vec{V}_\infty := e^{it|\nabla|} \vec{U}_\infty(0), \quad \vec{V}_n := e^{it\langle \nabla \rangle_n} \vec{U}_n(0), \quad \vec{v}_n = T_n \vec{V}_n(t/h_n).\tag{4.24}$$

For any $0 < R_n \rightarrow 0$ such that $h_n/R_n \rightarrow 0$, we have

$$\|\mathcal{F} \tilde{\gamma}_n\|_{L^\infty(0, S; L^2(|\xi| > R_n))} \lesssim \|\tilde{\gamma}_n\|_{L^\infty(0, S; L_x^2)} + o(1).\tag{4.25}$$

For the lower frequency, we have by the energy inequality, Hölder and Sobolev,

$$\begin{aligned}\|\vec{U}_n - \vec{V}_n\|_{L_t^\infty \dot{H}_x^{-1}(0, S)} &\lesssim \|f'(U_n)\|_{L_t^1 \dot{H}_x^{-1}(0, S)} \lesssim S \|U_n\|_{L_t^\infty \dot{H}_x^1(0, S)}^{p+1} \\ &\lesssim S (\|\vec{U}_\infty\|_{L_t^\infty L_x^2(0, S)} + \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2(0, S)})^{p+1},\end{aligned}\tag{4.26}$$

and similarly, $\|\vec{U}_\infty - \vec{V}_\infty\|_{L_t^\infty \dot{H}_x^{-1}(0, S)} \lesssim S \|\vec{U}_\infty\|_{L_t^\infty L_x^2}^{p+1}$. Since $|\langle \xi \rangle_n - \langle \xi \rangle_\infty| \leq h_n$, we have also $\|\vec{V}_n(t) - \vec{V}_\infty(t)\|_{L_x^2} \lesssim |t| h_n \|\vec{U}_\infty(0)\|_{L^2} + \delta$. Hence

$$\begin{aligned}\|\mathcal{F} \tilde{\gamma}_n\|_{L^\infty(0, S; L^2(|\xi| < R_n))} &\leq R_n \|\vec{U}_n - \vec{V}_n\|_{L_t^\infty \dot{H}_x^{-1}(0, S)} + \|\vec{V}_n - \vec{V}_\infty\|_{L_t^\infty L_x^2(0, S)} \\ &\quad + R_n \|\vec{V}_\infty - \vec{U}_\infty\|_{L_t^\infty \dot{H}_x^{-1}(0, S)} \\ &\lesssim o(1) S \|\tilde{\gamma}_n\|_{L_t^\infty L_x^2(0, S)}^{p+1} + \delta + o(1)\end{aligned}\tag{4.27}$$

Adding it to (4.25), we obtain

$$\|\vec{\gamma}_n\|_{L_t^\infty L_x^2(0,S)} \lesssim \|\vec{\gamma}_n\|_{L_t^\infty L_x^2(0,S)} + o(1)S\|\vec{\gamma}_n\|_{L_t^\infty L_x^2(0,S)}^{p+1} + \delta + o(1). \quad (4.28)$$

Combining it with the above estimates (4.23), we deduce that both $\vec{\gamma}_n$ and $\vec{\gamma}_n$ are bounded in $L_t^\infty L_x^2(0, S)$. \square

To prove (4.12) from the above claim, we decompose I into subintervals I_j , such that $\|U_\infty\|_{([W]_2^\bullet \cap [\widetilde{M}]_{2p}^\bullet)(I_j)} \leq \eta$ for each j . Then applying the above claim iteratively to the subintervals for small $\delta > 0$ yields (4.12), where the bound on $[M]_0$ is derived by interpolation and Sobolev embedding of $[H]_2^\bullet$ and $[W]_2^\bullet$.

For the estimate in $[W]_0$, we have by scaling

$$\begin{aligned} \|h_n T_n U_n((t - t_n)/h_n)\|_{[W]_0(h_n I + t_n)} &\sim h_n^{1/2} \|U_n\|_{[W]_0(I)} \\ &\lesssim h_n^{1/2} \|U_n\|_{[W]_2^\bullet(I)} + \|P_{<1} v_n\|_{[W]_0(I)} + h_n^{1/2} \|P_{<h_n}(U_n - V_n)\|_{[W]_0(I)}, \end{aligned} \quad (4.29)$$

where $\vec{V}_n := e^{it\langle \nabla \rangle_n} \vec{U}_n(0)$ and $\vec{v}_n = T_n \vec{V}_n(t/h_n)$. The first term on the right is vanishing since $\|U_n\|_{[W]_2^\bullet(I)}$ is bounded as shown above. The second term is $O(\delta)$ by Lemma 4.1. The third term is bounded, using Sobolev, Hölder and the same estimate as in (4.26), by

$$\begin{aligned} |I|^{W_1} h_n^{1/2+d(1/2-W_2)} \|U_n - V_n\|_{L_t^\infty L_x^2(I)} \\ \lesssim (|I| h_n)^{3/2-1/(d+1)} (\|\vec{U}_\infty\|_{L_t^\infty L_x^2(I)} + \varepsilon)^{p+1} = o(1), \end{aligned} \quad (4.30)$$

hence (4.29) is $O(\delta)$ for large n . This concludes the proof of the lemma for $d \geq 6$.

The case $d \leq 5$ is the same, but the nonlinear estimate is much simpler. In (4.14), $[\widetilde{M}]_{2p}^\bullet$ is replaced with $[M]_0$, and by the standard Strichartz, we have

$$\begin{aligned} \|\gamma_n\|_{[W]_2^\bullet \cap [M]_0} + \|\vec{\gamma}_n\|_{L_t^\infty L_x^2} \\ \lesssim \|\vec{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[W^{*(1)}]_2^\bullet} + \|h_n^2 U_n\|_{L_t^1 L_x^2}, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} \|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[W^{*(1)}]_2^\bullet} &\lesssim \|(U_\infty, \gamma_n)\|_{[W]_2^\bullet \cap [M]_0}^p \|\gamma_n\|_{[W]_2^\bullet \cap [M]_0} \\ &\lesssim (\eta + \|\gamma_n\|_{[W]_2^\bullet \cap [M]_0})^p \|\gamma_n\|_{[W]_2^\bullet \cap [M]_0}. \end{aligned} \quad (4.32)$$

Then estimating $\|h_n^2 U_n\|_{L_t^1 L_x^2(0,S)}$ in the same way as for $d \geq 6$, we obtain (4.15) without the last term. (4.29) is the same as above. \square

Proof of Theorem 3.1. Let v_n, V_n, V_∞ be the free solutions defined by

$$\vec{V}_n = e^{it\langle \nabla \rangle_n} \psi, \quad \vec{V}_\infty = e^{it|\nabla|} \psi, \quad \vec{v}_n = T_n V_n((t - t_n)/h_n), \quad (4.33)$$

and

$$M := \|U_\infty\|_{[W]_2^\bullet(J)}. \quad (4.34)$$

First consider the case $\tau_\infty = \infty$. Let $0 < \varepsilon < 1$ and choose $S > 0$ so large that

$$\delta_0 := \|V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S, \infty)} \leq \delta(\varepsilon, M), \quad (4.35)$$

where $\delta(\cdot, \cdot)$ is given by Lemma 4.2. Then Lemma 4.1 implies that

$$\|v_n\|_{([W]_2 \cap [M]_0)(h_n S + t_n, \infty)} \lesssim \delta_0 \quad (4.36)$$

for large n . If $\delta_0 \ll 1$, then the standard scattering argument for NLKG using the Strichartz norms implies that $u_{(n)}$ exists on $(h_n S + t_n, \infty)$, satisfying

$$\|\vec{u}_{(n)} - \vec{v}_n\|_{L_t^\infty L_x^2(h_n S + t_n, \infty)} + \|u_{(n)} - v_n\|_{([W]_2 \cap [M]_0)(h_n S + t_n, \infty)} \lesssim \delta_0^{2^*-1} \ll \delta_0, \quad (4.37)$$

and also for NLW

$$\|\vec{U}_\infty - \vec{V}_\infty\|_{L_t^\infty L_x^2(S, \infty)} + \|U_\infty - V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S, \infty)} \lesssim \delta_0^{2^*-1} \ll \delta_0. \quad (4.38)$$

Thus we obtain

$$\|u_{(n)}\|_{([W]_2 \cap [M]_0)(h_n S + t_n, \infty)} \lesssim \|V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S, \infty)} \sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(S, \infty)}, \quad (4.39)$$

and, for large n ,

$$\|\vec{U}_{(n)}(S) - \vec{V}_n(S)\|_{L_x^2} + \|\vec{V}_n(S) - \vec{V}_\infty(S)\|_{L_x^2} + \|\vec{V}_\infty(S) - \vec{U}_\infty(S)\|_{L_x^2} \ll \delta_0. \quad (4.40)$$

The next step is to go from S to the negative time direction. If J is bounded from below, then let $S' := \inf J$. Otherwise, choose $S' < S$ so that

$$\|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(-\infty, S')} < \varepsilon. \quad (4.41)$$

Applying Lemma 4.2 to U_∞ and $U_{(n)}$ backward in time from $t = S$, we obtain

$$\|\vec{U}_{(n)} - \vec{U}_\infty\|_{L_t^\infty L_x^2(S', S)} + \|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(S', S)} < \varepsilon, \quad (4.42)$$

and $\|u_{(n)}\|_{[W]_0(h_n S' + t_n, h_n S + t_n)} \lesssim \delta_0$ for large n .

If J is unbounded from below, we have still to go from S' to $-\infty$. The standard argument for small data scattering of NLW for $t \rightarrow -\infty$ implies that

$$\|\operatorname{Re} |\nabla|^{-1} e^{it|\nabla|} \vec{U}_\infty(S')\|_{([W]_2^\bullet \cap [M]_0)(-\infty, 0)} \sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(-\infty, S')} < \varepsilon. \quad (4.43)$$

Then Lemma 4.1 applied backward in t implies for large n

$$\|\operatorname{Re} \langle \nabla \rangle^{-1} e^{it\langle \nabla \rangle} T_n \vec{U}_\infty(S')\|_{([W]_2 \cap [M]_0)(-\infty, 0)} \lesssim \varepsilon. \quad (4.44)$$

Let w_n be the solution of NLKG with $\vec{w}_n(0) = T_n \vec{U}_{(n)}(S')$. Then the above estimate together with $\|\vec{U}_{(n)}(S') - \vec{U}_\infty(S')\|_{L_x^2} < \varepsilon$ and the scattering for NLKG implies

$$\|w_n\|_{([W]_2 \cap [M]_0)(-\infty, 0)} \lesssim \varepsilon. \quad (4.45)$$

Since $w_n = h_n T_n U_{(n)}(t/h_n + S') = u_{(n)}(t + h_n S' + t_n)$, we deduce that

$$\begin{aligned} \|U_{(n)}\|_{([W]_2^\bullet \cap [M]_0)(-\infty, S')} &\sim \|u_{(n)}\|_{([W]_2^\bullet \cap [M]_0)(-\infty, h_n S' + t_n)} \\ &\lesssim \|u_{(n)}\|_{([W]_2 \cap [M]_0)(-\infty, h_n S' + t_n)} = \|w_n\|_{([W]_2 \cap [M]_0)(-\infty, 0)} \lesssim \varepsilon. \end{aligned} \quad (4.46)$$

Thus we obtain, in the case $\tau_\infty = \infty$,

$$\|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + \|u_{(n)}\|_{[W]_0(h_n J + t_n)} \lesssim \varepsilon + \delta_0 \quad (4.47)$$

for large n . Since ε and δ_0 can be chosen as small as we wish, it implies

$$\lim_{n \rightarrow \infty} \|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + \|u_{(n)}\|_{[W]_0(h_n J + t_n)} = 0, \quad (4.48)$$

and by scaling,

$$\begin{aligned} \|u_{(n)}\|_{([W]_2 \cap [M]_0)(h_n J + t_n)} &\sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + \|u_{(n)}\|_{[W]_0(h_n J + t_n)} \\ &= \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(J)} + o(1). \end{aligned} \quad (4.49)$$

Since $S \rightarrow \infty$ and $S' \rightarrow \inf J$ as $\varepsilon, \delta \rightarrow +0$, we also obtain

$$\lim_{n \rightarrow \infty} \|\vec{U}_{(n)} - \vec{U}_\infty\|_{L_t^\infty L_x^2(I)} = 0, \quad (4.50)$$

for any finite subinterval I . The case $\tau_\infty = -\infty$ is the same by the time symmetry.

If $\tau_\infty \in \mathbb{R}$, then $\|\vec{U}_{(n)}(\tau_\infty) - \vec{U}_\infty(\tau_\infty)\|_{L_x^2} \rightarrow 0$. Hence the same argument as we used above to go from S to $-\infty$ yields

$$0 = \lim_{n \rightarrow \infty} \|\vec{U}_{(n)} - \vec{U}_\infty\|_{L_t^\infty L_x^2(S', \tau_\infty)} = \lim_{n \rightarrow \infty} \|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(\inf J, \tau_\infty)}, \quad (4.51)$$

for any $S' \in (\inf J, \tau_\infty)$, and also on $(\tau_\infty, \sup J)$ by the time symmetry. Thus we obtain (4.48) and (4.50) for any $\tau_\infty \in [-\infty, \infty]$. \square

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REFERENCES

- [1] Slim Ibrahim, Nader Masmoudi and Kenji Nakanishi, *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, Analysis & PDE **4** (2011) no. 3, 405–460.
- [2] Slim Ibrahim, Nader Masmoudi and Kenji Nakanishi, *Threshold solutions in the case of mass-shift for the critical Klein-Gordon equation*, Trans. Amer. Math. Soc. **366** (2014), 5653–5669.
- [3] Slim Ibrahim, Nader Masmoudi and Kenji Nakanishi, *Trudinger-Moser inequality on the whole plane with the exact growth condition*, J. Eur. Math. Soc. **17** (2015), 819–835.

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